The H-polynomial of a Group Embedding[*]

Lex E. Renner

May 2008

Abstract

The Poincaré polynomial of a Weyl group calculates the Betti numbers of the projective homogeneous space G/B, while the h-vector of a simple polytope calculates the Betti numbers of the corresponding rationally smooth toric variety. There is a common generalization of these two extremes called the H-polynomial. It applies to projective, homogeneous spaces, toric varieties and, much more generally, to any algebraic variety X where there is a connected, solvable, algebraic group acting with a finite number of orbits. We illustrate this situation by describing the H-polynomials of certain projective $G \times G$ - varieties X, where G is a semisimple group and B is a Borel subgroup of G. This description is made possible by finding an appropriate cellular decomposition for X and then describing the cells combinatorially in terms of the underlying monoid of $B \times B$ - orbits. The most familiar example here is the wonderful compactification of a semisimple group of adjoint type.

1 Introduction

Let G_0 be a semisimple algebraic group and let $\rho : G_0 \to End(V)$ be a representation of G_0. Define Y_ρ to be the Zariski closure of $G = [\rho(G_0)] \subseteq \mathbb{P}(End(V))$, the projective space associated with $End(V)$. Finally, let X_ρ be the normalization of Y_ρ. X_ρ is a projective, normal $G \times G$-embedding of G. That is, there is an open embedding $G \subseteq X_\rho$ such that the action $G \times G \times G \to G$, $(g, h, x) \rightsquigarrow gxh^{-1}$, extends over X_ρ. Furthermore $B \times B$ acts on X_ρ with a finite number of orbits. See [1] for an up-to-date description of these (and other) embeddings. The problem here is to find a homologically useful description of how X_ρ fits together from these $B \times B$-orbits. This has been accomplished by several authors in case X is the wonderful embedding of an adjoint semisimple group. See [4, 9, 17, 25] for an assortment of approaches. The purpose of this survey is to describe what can be done here for any rationally smooth embedding of the form X_ρ.

There is very little mystery in stating the problem. Suppose we have a rationally smooth, embedding X_ρ. We wish to describe and calculate the Betti numbers of X_ρ in terms of the

---

[*]This is the text of a talk given at "A Conference in Algebraic Geometry, Honoring F. Hirzebruch 80th birthday, May 18-23, 2008, Bar Ilan University, Israel".

$B \times B$-orbit structure of X_ρ. Our approach is based on unraveling the so called H-polynomial of X_ρ (see Section 2 below). This H-polynomial is the primary combinatorial invariant of a group embedding. The challenge here is to first organize the $B \times B$-orbits into "rational" cells, and then to quantify these cells in terms of salient Weyl group data and toric data. The overarching fact here is Theorem 2.5 of [21] whereby we characterize the condition "X_ρ is rationally smooth" in terms of related toric data. This allows us to obtain a homologically meaningful decomposition of X_ρ into rational cells. The idempotent system of the associated monoid cone $M_\rho \subseteq End(V)$ then helps us calculate the dimension of each cell.

The H-polynomial is depicted as a synthesis of the h-polynomial from toric geometry, and the length polynomial from the theory of projective homogeneous spaces. The purpose of the H-polynomial is to encode topological information about X_ρ by organizing numerical and combinatorial properties of the $B \times B$-orbits on X_ρ.

Throughout this survey we omit the proofs of many statements. The interested reader should consult [19, 20, 21, 22, 23] for more details.

2 H-polynomials

One can associate with a smooth torus embedding its h-**polynomial**. And with a projective homogeneous space (or Schubert variety) we can associate its **length polynomial**. In both cases we obtain a polynomial that can be used to calculate Betti numbers. In more general situations, like $G \times G$-embeddings, we need to define a synthesis of these two extremes. This leads us naturally to the notion of an H-**polynomial**, which is the obvious synthesis of the h-polynomial and the length polynomial.

2.1   Length Polynomials

Let (W, S) be a Weyl group with length function $l : W \to \mathbb{N}$. Define the **length Polynomial** $P_W(t)$ of W by setting

$$P_W(t) = \sum_{w \in W} t^{l(w)}.$$

It is well-known, and much-studied, that $P_W(t)$ can be used to calculate the Betti numbers of G/B. This generalizes to the other projective homogeneous spaces. If $P \subseteq G$ is a parabolic subgroup then $P = P_J$ for some unique $J \subseteq S$. We then define the length polynomial of W^J by setting

$$P_{W^J}(t) = \sum_{w \in W^J} t^{l(w)}.$$

Here W^J is the set of minimal coset representatives of W_J in W.

For example consider S_4, the symmetric group on four letters. Its canonical generators are the three transpositions $r = (1, 2)$, $s = (2, 3)$ and $t = (3, 4)$. It is easy to calculate that

$$|\{x \in S_4 \mid l(x) = 0\}| = 1,$$

$$|\{x \in S_4 \mid l(x) = 1\}| = 3,$$

$$|\{x \in S_4 \mid l(x) = 2\}| = 5,$$

$$|\{x \in S_4 \mid l(x) = 3\}| = 6,$$

$$|\{x \in S_4 \mid l(x) = 4\}| = 5,$$

$$|\{x \in S_4 \mid l(x) = 5\}| = 3 \text{ and}$$

$$|\{x \in S_4 \mid l(x) = 6\}| = 1.$$

Thus the Poincaré polynomial of the flag variety $\mathcal{F}_4(\mathbb{C})$ is

$$P(t) = 1 + 3t^2 + 5t^4 + 6t^6 + 5t^8 + 3t^{10} + t^{12}.$$

Similar formulas hold for the Grassmanians and other projective homogeneous spaces.

2.2 h-polynomials

Let X be a projective torus embedding. Define the h-**polynomial** of X by setting

$$h_X(t) = \sum_{Z < X} (t - 1)^{dim(Z)}$$

where $Z < X$ means that Z is a T-orbit of X. In case X is rationally smooth ([5, 8, 10]) $h_X(t)$ can be used to calculate the Betti numbers of X.

For example consider the **Eulerian Polynomials**. Let S_n denote the symmetric group, and let $\sigma = (p_1 p_2 ... p_n) \in S_n$ (so that $\sigma(i) = p_i$). Define

$$A(\sigma) = \{i \mid 1 \le i \le n - 1 \text{ and } p_i \le p_{i+1}\},$$

the **ascent set** of σ. For example, in S_4,

$$A(1234) = \{1, 2, 3\}$$

$$A(1324) = \{1, 3\}$$

$$A(4321) = \phi$$

Define

$$a(\sigma) = |A(\sigma)|,$$

and

$$E_n(t) = \sum_{\sigma \in S_n} t^{a(\sigma)}.$$

E_n is the Eulerian polynomial for S_n. This is the h-polynomial of the torus embedding X_{n-1} associated (via the normal fan construction) with the much-studied **permutahedron** $\mathcal{P}_{n-1}$. $\mathcal{P}_{n-1}$ can be defined (ambiguously) as the convex hull of the S_n-orbit of a point in general position in $\mathbb{Q}^n$. To prove that $E_n(t)$ is the h-polynomial of X_{n-1} one can use the method of

3

Bialynicki-Birula [2] to obtain a cellular decomposition of X_{n-1} with one cell of dimension $a(\sigma)$ for each $\sigma \in S_n$. The face lattice $\mathcal{F}$ of $\mathcal{P}_{n-1}$ can be identified with

$$\bigsqcup_{\sigma \in S_n} \{(\sigma, A) \mid A \subseteq A(\sigma)\}.$$

Thus the h-polynomial of X_{n-1} is

$$h_n(t) = \sum_{\sigma \in S_n, \ A \subseteq A(\sigma)} (t-1)^{|A|}.$$

But

$$t^{a(\sigma)} = ((t-1)+1)^{a(\sigma)} = \sum_{A \subseteq A(\sigma)} (t-1)^{|A|}.$$

Thus $E_n(t) = h_n(t)$.

The following illustration shows us the ascent structure for the case S_4. It is a useful, but elementary exercise to calculate this polynomial by looking at the picture. One obtains

$$E_4(t) = 1 + 11t + 11t^2 + t^3.$$

Notice that the set of edges of the polytope $\mathcal{P}_{n-1}$ is canonically identified with

$$\bigsqcup_{\sigma \in S_n} \{(\sigma, \alpha) \mid \alpha \in A(\sigma)\}.$$

This set is also identified with the set of one-dimensional T-orbits on X_{n-1}.
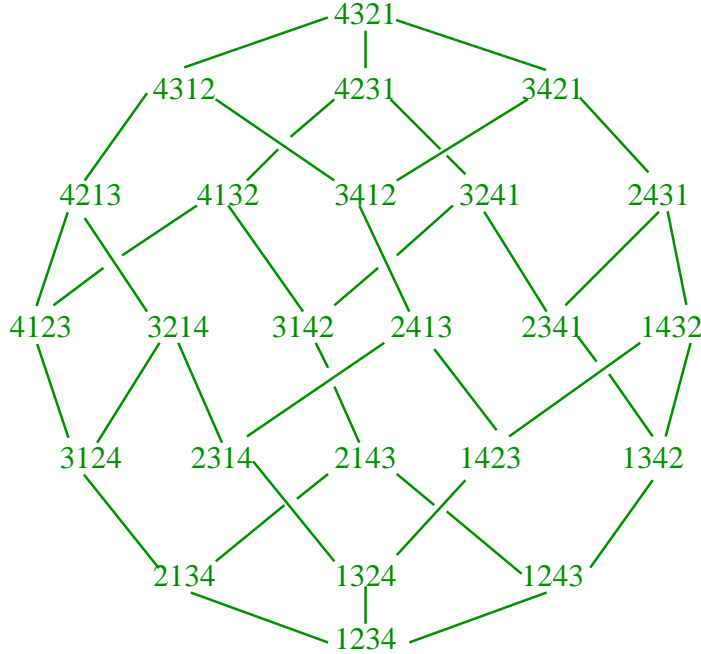
2.3 H-polynomials

The H-polynomial is the obvious synthesis of two extremes, the h-polynomial of a torus embedding, and the length polynomial of a Weyl group. In the former case one collects summands of the form $(t-1)^a$ (coming from the finite number of orbits of a torus group) while in the latter case one collects summands of the form t^b (coming from the finite number of orbits of a unipotent group). But in each case the corresponding polynomial yields the desired coefficients. The common theme here is that, in both cases, we are summing over a finite number of H-orbits for the appropriate solvable group H. In more general cases, like $G \times G$-embeddings of G with the $B \times B$-action, there are a finite number of $B \times B$-orbits, and each one is composed of a unipotent part and a diagonalizable part. In this situation, we need to collect summands of the form $(t-1)^a t^b$ for the appropriate integers a and b. Indeed, for each $B \times B$-orbit BxB, define

$$a(x) = rank(B \times B) - rank(B \times B)_x$$

and

$$b(x) = dim(UxU).$$

Here $(B \times B)_x = \{(g,h) \in B \times B \mid gxh^{-1} = x\}$. Thus we make the following fundamental definition.

Definition 2.1. Let $\rho : G \to End(V)$ be an irreducible representation and let

$$X = X_\rho$$

be as above. The **H-polynomial** of X is defined to be

$$H_X(t) = \sum_{x \in \mathcal{R}} (t-1)^{a(x)} t^{b(x)}.$$

where $\mathcal{R}$ is a set of representatives for the $B \times B$-orbits of X.

Remark 2.2. This H-polynomial is not the correct tool for investigating varieties with singularities that are not rationally smooth. In the case of Schubert varieties, and Kazhdan-Lusztig theory, the correct formulation incorporates a "correction factor" (*aka* the **KL-polynomial**) that takes into account local intersection cohomology groups. See Theorem 6.2.10 of [3].

The authors of [7] calculate the Poincaré polynomial, for intersection cohomology, of a large class of $G \times G$-embeddings using the stratification by $G \times G$-orbits. In case the singularities of $\mathbb{P}(M)$ are rationally smooth, the polynomial $IP_X(t)$ of [7] agrees with the polynomial $H_X(t)$ (where $X = \mathbb{P}(M)$) defined above in Definition 2.1. See Theorem 3.5. However, in the absence of rationally smooth singularities, these local intersection cohomology groups may not be so well adapted to cellular decompositions.

See [25] for a detailed study of the intersection cohomology of $B \times B$-orbit closures in the case of the "wonderful embedding" (i.e. When M is $\mathcal{J}$-irreducible of type $J = \emptyset$ and $X = (M \setminus \{0\})/\mathbb{C}^*$. Even though $M \setminus \{0\}$ is rationally smooth in this case, the same may not true for the closure in $M \setminus \{0\}$, of a $B \times B$-orbit).

Example 2.3. Let $M = M_2(K)$. Then

$$\mathcal{R} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}.$$

For simplicity we write it, in order, as $\mathcal{R} = \{1, s, e, f, n, m, 0\}$. It is easy to calculate $a(x)$ and $b(x)$ for each element $x \in \mathcal{R}$. For example, $a(m) = 1$ and $b(m) = 2$. Thus, if $X = M_2(\mathbb{C})$,

$$H_X(t) = (t-1)^2 t^1 + (t-1)^2 t^2 + (t-1)^1 t^1 + (t-1)^1 t^1 + (t-1)^1 t^0 + (t-1)^1 t^2 + (t-1)^0 t^0$$

Hence, after an elementary calculation, we obtain

$$H_X(t) = t^4.$$

This should not be surprising since, over $\mathbb{F}_q$,

$$|BxB| = (q-1)^{a(x)} q^{b(x)}.$$

while

$$M_2(\mathbb{C}) = \bigsqcup_{x \in \mathcal{R}} BxB.$$

Here, B is the 2×2 upper-triangular group.

Example 2.4. Let $G_0 = PGL_3(\mathbb{C})$, and let $\rho : G_0 \to End(V)$ be any irreducible representation whose highest weight is in general position. Then the H-polynomial of X_ρ is given by

$$H(t) = \left[1 + 2t^2 + 2t^3 + t^5\right]\left[1 + 2t + 2t^2 + t^3\right]$$

See §6.2 and Theorem 4.2 for some related general formulas.

3 Rationally Smooth Embeddings

Let X be a complex, algebraic variety of dimension n. Then X is **rationally smooth** at $x \in X$ if there is a neighborhood U of x in the complex topology such that, for any $y \in U$,

$$H^m(X, X \setminus \{y\}) = (0)$$

for $m \neq 2n$ and

$$H^{2n}(X, X \setminus \{y\}) = \mathbb{Q}.$$

Here $H^*(X)$ denotes the cohomology of X with rational coefficients. Danilov, in [8], characterized the rationally smooth toric varieties in combinatorial terms.

The purpose of this section is to identify the class of rationally smooth embeddings of the form X_ρ. We then find a useful description of the H-polynomial of these embeddings. See Theorem 4.2.

3.1 Rationally Smooth Monoids

Let M and N be reductive monoids. We write $M \sim_0 N$ if there is a reductive monoid L and finite dominant morphisms $L \to M$ and $L \to N$ of algebraic monoids. One can check that this is indeed an equivalence relation.

In the following theorem we write $\mathbb{C}^n$ for the reductive monoid with multiplication

$$m : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$$

defined by $m((t_1, ...t_n), (x_1, ...x_n)) = (t_1 x_1, ...t_n x_n)$.

Theorem 3.1. *Let M be a reductive monoid with zero. The following are equivalent.*

1. *M is rationally smooth.*

2. *$M \sim_0 \Pi_i M_{n_i}(\mathbb{C})$.*

3. *$\overline{T} \sim_0 \mathbb{C}^m$.*

Notice that if $M \sim_0 N$ then M is rationally smooth if and only if N is rationally smooth.

See [21] for a proof of Theorem 3.1. See also [5] for a systematic discussion of rationally smooth varieties with torus action.

Corollary 3.2. *Let M be a rationally smooth reductive monoid and let $e \in E(M)$. Then eM is a rationally smooth algebraic variety.*

Proof. Let T be a maximal torus with $e \in \overline{T}$. Then there is a one-parameter subgroup $S \subseteq T$ such that $E(\overline{S}) = \{1, e\}$. One checks easily that $eM = \{x \in M \mid sx = x \text{ for all } s \in S\}$. Thus, by Theorem 1.1 of [5], eM is rationally smooth. $\square$

In [19] it is shown that any rationally smooth embedding of the form X_ρ (see §3.2 below) has a natural cell decomposition. Corollary 3.2 above allows one to conclude that these cells are themselves rationally smooth. See Theorem 5.1 of [19].

3.2 Rationally Smooth Embeddings

Let M be a normal, reductive monoid with unit group G and zero element $0 \in M$. Let $\epsilon : \mathbb{C}^* \to G$ be a central 1-parameter subgroup that converges to 0. Define

$$\mathbb{P}_\epsilon(M) = (M \setminus \{0\})/\mathbb{C}^*.$$

$\mathbb{P}_\epsilon(M)$ is a projective $G \times G$-embedding. $G \times G$ acts on $\mathbb{P}_\epsilon(M)$ by

$$(g, h, [x]) \rightsquigarrow [gxh^{-1}].$$

If M is semisimple (so that ϵ is essentially unique) we write $\mathbb{P}(M)$ for $\mathbb{P}_\epsilon(M)$. Notice also that there is a representation ρ of G such that

$$\mathbb{P}_\epsilon(M) = X_\rho.$$

7

Recall that $E(M) = \{e \in M \mid e^2 = e\}$. For $e \in E(M)$ we define

$$M_e = \overline{\{g \in G \mid ge = eg = e\}}.$$

By the results of [6], M_e is an irreducible, normal reductive monoid with unit group $G_e = \{g \in G \mid ge = eg = e\}$.

Theorem 3.3. *The following are equivalent.*

1. *$\mathbb{P}_\epsilon(M)$ is rationally smooth.*

2. *For any $e \in E(M) \setminus \{0\}$, M_e is rationally smooth.*

Indeed, by the results of [6, 16], $M \setminus \{0\}$ is locally isomorphic to the product of M_e ($e \in E_1$) with some affine space. Thus $\mathbb{P}_\epsilon(M)$ is rationally smooth if and only if so is M_e for all $e \in E_1$.

Notice, in particular, that the condition "$\mathbb{P}_\epsilon(M)$ is rationally smooth" is independent of how we choose ϵ.

3.3 H-polynomials and Poincaré Polynomials

Definition 3.4. Let X be a complex algebraic variety. The **Poincaré polynomial** of X is the polynomial $P_X(t)$ with the signed Betti numbers of X as coefficients.

$$P_X(t) = \sum_{i \geq 0} (-1)^i dim_{\mathbb{Q}}[H^i(X; \mathbb{Q})] t^i.$$

Clearly one can define a Poincaré polynomial for any reasonable cohomology theory. In [7] the authors compute the **intersection cohomology** Poincaré polynomial $IP_X(t)$ for a large class of $G \times G$-embeddings X of G. However it is known ([10]) that $IP_X(t) = P_X(t)$ in case X has rationally smooth singularities.

Theorem 3.5. *Let M be a semisimple algebraic monoid such that $M \setminus \{0\}$ is rationally smooth. Then*

$$H_X(t^2) = P_X(t)$$

where $X = [M \setminus \{0\}]/K^*$.

Proof. We give a sketch. See [23] for more details. By our assumptions on M, X is rationally smooth. Hence by the results of McCrory in [10], $IP_X(t) = P_X(t)$. Hence it suffices to show that $H(M)(t^2) = IP_X(t)$.

Let $x \in X$. Then from Theorem 1.1 of [7]

$$IP_{X,x}(t) = \tau_{\leq d_x - 1}((1 - t^2) IP_{\mathbb{P}(S_x)}(t))$$

where S_x is the appropriate slice and $d_x = dim(S_x)$. One checks, using Lemma 1.3 of [5] and the results of [6] , that $\mathbb{P}(S_x)$ is a rational homology projective space of dimension $d_x - 1$.

Thus $IP_{X,x}(t) = 1$. Consequently, the formula for $IP_X(t)$ in Theorem 1.1 of [7] simplifies to a summation with summands of the form $P_{(G \times G)x}(t)$, as in (5.1.5) of [7]. Hence

$$IP_X(t) = \sum_x P_{(G \times G)x}(t),$$

where the sum is taken over a set of representatives of the $G \times G$-orbits of X. But this is the same formula that one obtains by combining the $B \times B$-orbits into one summand for each $G \times G$-orbit in the formula for $H(M)(t^2)$. $\qquad\qquad\square$

4   Cellular Decompositions and H-polynomials

Assume that X_ρ is rationally smooth. We then obtain a kind of cellular decomposition for X_ρ. This decomposition is ultimately controlled by the idempotent system of the associated monoid cone M of X_ρ. See Theorem 4.1 below. This allows us to find a useful way to label these cells in terms of quantities from W and $E(\overline{T})$. We then find a dimension formula for each cell in terms of its label. Finally we find an appropriate homological interpretation of these cells so that we can compute the Betti numbers of X_ρ in terms of the H-polynomial. See also Theorem 3.5.

4.1   Monoid BB-decompositions

In this section we study the BB-cells of $\mathbb{P}_\epsilon(M) = (M \backslash \{0\})/Z$ where M is a reductive monoid with zero and $\epsilon : Z \subset G$ is a central one-parameter subgroup with $0 \in \overline{Z}$.

Let X be a normal, projective variety and assume that $S = K^*$ acts on X. If $F_i \subset X^S$ is a connected component of the fixed point set X^S we define, following [2],

$$X_i = \{x \in X \mid \lim_{t \to 0}(t \cdot x) \in F_i\}.$$

This decomposes X as a disjoint union

$$X = \bigsqcup_i X_i$$

of locally closed subsets. Furthermore we have the BB-maps

$$\pi_i : X_i \to F_i$$

defined by $\pi_i(x) = \lim_{t \to 0}(t \cdot x)$. See [2] for more details. In that paper the author assumes that X is nonsingular. Then he proves his much-celebrated results (see Theorem 4.3 of [2]). However many of his ideas can be extended to the nonsingular case. On the other hand, the purpose of our discussion is to describe this BB-**decomposition** in terms of the system of idempotents of an appropriate algebraic monoid.

We start with the following results from [19]. These theorems are stated in [19] for semisimple monoids, but the proofs are easily modified to apply to all reductive monoids

with zero. Let $\overline{T}$ be the closure in M of a maximal torus, and let $E(\overline{T})$ be the set of idempotents of $\overline{T}$. Let

$$E_1 = E_1(\overline{T}) = \{e \in \overline{T} \mid dim(Te) = 1\},$$

the set of rank-one idempotents of $\overline{T}$.

Theorem 4.1. *Let M be a reductive monoid with unit group G, zero element $0 \in M$, Borel subgroup $B \subseteq G$ and maximal torus $T \subseteq B$. Let $\epsilon : Z \subseteq G$ be a 1-dimensional, connected, central subgroup of G such that $0 \in \overline{Z}$. Choose a one parameter subgroup $\lambda : \mathbb{C}^* \to T$ such that*

1. $\lim\limits_{t \to 0}(tut^{-1}) = 1$ for all $u \in B_u$.

2. $\{x \in M\backslash\{0\} \mid \lambda(t)x \in Zx$ for all $t \in \mathbb{C}^*\} = \bigcup_{e \in E_1(\overline{T})} eM$.

Let $X = (M\backslash\{0\})/Z = \mathbb{P}_\epsilon(M)$ and let

$$X = \bigsqcup_{e \in E_1} X(e)$$

be the BB-decomposition of X with respect to λ. Then, for all $e \in E_1(\overline{T})$,

$$X(e) \subseteq \{[y] \in X \mid eBy = eBey \subseteq eG\}.$$

Furthermore, the BB projection,

$$\pi_e : X(e) \to (eG)/\mathbb{C}^* \cong G/P_e$$

is given by $\pi_e([y]) = [ey]$.

Theorem 4.1 says that, if we choose an appropriate one-parameter-subgroup of T, the BB-projections can be calculated by using the rank-one idempotents of $\overline{T}$. Since we assume that $\mathbb{P}_\epsilon(M) = X_\rho$ is rationally smooth, the fibre of each π_e is a certain "rational cell" that has been computed in [19]. On the other hand, G/P_e has an easily quantified cell decomposition

$$G/P_e = \sqcup A_w$$

Thus, if we let $\pi_e^{-1}(A_w) = C_w$, we obtain the decomposition $X_e = \sqcup_w C_w$. From Corollary 5.2 of [19] each cell C has an H-polynomial of the form

$$H(C) = \sum_{r \in \mathcal{R} \cap C} t^{a(r)}(t-1)^{b(r)} = t^{dim(C)}.$$

Putting this all together, we can calculate the H-polynomial of $\mathbb{P}_\epsilon(M)$ by identifying the contribution of each C and each $X(e)$.

To state the main theorem we first recall that if (W, S) is a Weyl group and $J \subset S$ then W^J is the set of minimal length representatives for the cosets of W_J in W. In particular, the canonical composition

$$W^J \to W \to W/W_J$$

is bijective.

10

Theorem 4.2. *Let M be a reductive monoid such that $\mathbb{P}_\epsilon(M)$ is rationally smooth, and let $\mathcal{R}$ be the monoid of $B \times B$-orbits of M. Write G/P_e for $(eG)/\mathbb{C}^*$. Let $e_1 \in \Lambda_1$ and let $we_1w^{-1} = e$, where $w \in W^J$. $w_0 \in W^J$ is the unique element of maximal length, and $J = \{s \in S \mid se_1 = e_1s\}$. Also, let $H(G/P_e) = \sum_{w \in W^J} t^{l(w)}$. Finally, $\nu(e)$ is defined in Definition 5.4 of [19].*

1. *If we let $w(e) = w$ then*
$$H_{\mathbb{P}_\epsilon(M)}(t) = \sum_{e \in E_1} \left[t^{l(w_0)-l(w(e))+\nu(e)} H(G/P_e) \right].$$

2. *In case P_e and $P_{e'}$ are conjugate for all $e, e' \in E_1$ the sum can be rewritten as*
$$H_{\mathbb{P}_\epsilon(M)}(t) = \left[\sum_{e \in E_1} t^{l(w_0)-l(w(e))+\nu(e)} \right] H(G/P).$$

   where $P = P_e$.

See Theorem 5.5 of [19].

Remark 4.3. $\nu(e)$ is the most difficult quantity to calculate in the above setup. It contains a subtle contribution from the induced BB-decomposition of the associated maximal torus. See Sections 4.2 and 5.1 of [19] for more details. In some examples this calculation involves **descent systems** [20]. See Section 5 for a discussion of these descent systems, and see Section 6.2 below (the wonderful embedding) for the motivating example. See Theorem 6.5 and Examples 6.6, 6.7 and 6.8 for more illustration of how $\nu(e)$ is quantified in the case of a **simple** embedding of the form X_ρ.

Each summand in the formula for $H_{\mathbb{P}(M)}(t)$ (in Theorem 4.2) mirrors the BB-projection
$$\pi_e : X(e) \to G/P_e.$$

In particular, the fibre of π_e has dimension $l(w_0) - l(w(e)) + \nu(e)$.

5 Descent Systems

As mentioned above in Remark 4.3, a major problem in calculating the H-polynomial is finding a satisfying description of the quantity $\nu(e)$ of Theorem 4.2. We are particularly interested in the situation where the embedding X_ρ is obtained from an irreducible representation $\rho = \rho_\lambda$, of type $J = \{s \in S \mid s(\lambda) = \lambda\}$, of G. We refer to this embedding as $\mathbb{P}(J)$. This is well-defined since $\mathbb{P}(J)$ depends only on J. This leads us to the notion of a **descent system** [20]. This descent system serves as an effective combinatorial substitute for the infinitesimal part of Bialynicki-Birula's method [2].

Definition 5.1. Let (W, S) be a Weyl group and let $J \subseteq S$ be a proper subset. Define
$$S^J = (W_J(S \setminus J)W_J) \cap W^J.$$

We refer to (W^J, S^J) as the **descent system** associated with $J \subseteq S$.

Proposition 5.2. *Let $u, v \in W^J$ be such that $u^{-1}v \in S^J W_J$. In particular, $u \neq v$. Then either $u < v$ or $v < u$ in the Bruhat order $<$ on W^J.*

For a proof see [20].
We let
$$S_s^J = W_J s W_J \cap W^J.$$

Definition 5.3. Let $w \in W^J$. Define

1. $D_s^J(w) = \{r \in S_s^J \mid wrc < w \text{ for some } c \in W_J\}$, and

2. $A_s^J(w) = \{r \in S_s^J \mid w < wr\}$.

We refer to $D^J(w) = \bigsqcup_{s \in S \setminus J} D_s^J(w)$ as the **descent set** of w relative to J, and $A^J(w) = \bigsqcup_{s \in S \setminus J} A_s^J(w)$ as the **ascent set** of w relative to J.

By Proposition 5.2, for any $w \in W^J$, $S^J = D^J(w) \sqcup A^J(w)$.

Remark 5.4. Notice that $wrc < w$ for some $c \in W_J$ if and only if $(wr)_0 < w$, where $(wr)_0 \in wrW_J$ is the element of minimal length in wrW_J. It is useful to illustrate the fact that $S^J = D^J(w) \sqcup A^J(w)$, for each $w \in W^J$, by doing some specific calculations.

Definition 5.5. For each $w \in W^J$ and each $s \in S \setminus J$ define $\nu_s(w) = |A_s^J(w)|$. We refer to $(W^J, \leq, \{\nu_s\})$ as the **augmented poset** of J. For convenience we let

$$\nu(w) = \sum_{s \in S \setminus J} \nu_s(w).$$

The point here is this. We use $(W^J, \leq, \{\nu_s\})$ to quantify how the underlying torus embedding of $\mathbb{P}(J)$ is involved in calculating the H-polynomial of $\mathbb{P}(J)$.

Example 5.6. Let
$$W = < s_1, ...s_n >$$
be the Weyl group of type A_n (so that $W \cong S_{n+1}$), and let

$$J = \{s_3, ...s_n\} \subseteq S.$$

If $w \in W^J$ then $w = a_p$, $w = b_q$, or else $w = a_p b_q$. Here $a_p = s_p \cdots s_1$ $(1 \leq p \leq n)$ and $b_q = s_q \cdots s_2$ $(2 \leq q \leq n)$. If we adopt the useful convention $a_0 = 1$ and $b_1 = 1$, then we can write
$$W^J = \{a_p b_q \mid 0 \leq p \leq n \text{ and } 1 \leq q \leq n\}$$
with uniqueness of decomposition. Let $w = a_p b_q \in W^J$. After some tedious calculation with braid relations and reflections, we obtain that

a) $A_{s_1}^J(a_p b_q) = \{s_1\}$ if $p < q$.
$\quad A_{s_1}^J(a_p b_q) = \emptyset$ if $q \leq p$.
$\quad$ Thus $\nu_{s_1}(a_p b_q) = 1$ if $p < q$ and $\nu_{s_1}(a_p b_q) = 0$ if $q \leq p$.

b) $A^J_{s_2}(a_p b_q) = \{s_m \cdots s_n \mid m > q\}$ if $q < n$.
   $A^J_{s_2}(a_p b_q) = \emptyset$ if $q = n$.
   Thus $\nu_{s_2}(a_p b_q) = n - q$.

It is interesting to compute the two-parameter "Euler polynomial"

$$H(t_1, t_2) = \sum_{w \in W^J} t_1^{\nu_1(w)} t_2^{\nu_2(w)}$$

of the augmented poset $(W^J, \leq, \{\nu_1, \nu_2\})$ (where we write ν_i for ν_{s_i}). A simple calculation yields

$$H(t_1, t_2) = \sum_{k=1}^{n} [kt_1 + (n + 1 - k)] t_2^{n-k}.$$

Let $r : W \to GL(V)$ be the usual reflection representation of the Weyl group W, where V is a rational vector space. Along with this goes the **Weyl chamber** $\mathcal{C} \subseteq V$ and the corresponding set of **simple reflections** $S \subseteq W$. The Weyl group W is generated by S, and $\mathcal{C}$ is a fundamental domain for the action of W on V.

Let $\lambda \in \mathcal{C}$. Consider the face lattice $\mathcal{F}_\lambda$ of the polytope

$$\mathcal{P}_\lambda = Conv(W \cdot \lambda),$$

the convex hull of $W \cdot \lambda$ in V. This lattice $\mathcal{F}_\lambda$ depends only on $W_\lambda = \{w \in W \mid w(\lambda) = \lambda\} = W_J = \langle s \mid s \in J \rangle$, where $J = \{s \in S \mid s(\lambda) = \lambda\}$. One can describe $\mathcal{F}_\lambda = \mathcal{F}_J$ explicitly in terms of $J \subseteq S$. See [12].

Definition 5.7. We refer to J as **combinatorially smooth** if $\mathcal{P}_\lambda$ is a simple polytope.

It is important to characterize the very interesting condition of Definition 5.7. If $J \subseteq S$ we let $\pi_0(J)$ denote the set of connected components of J. To be more precise, let $s, t \in J$. Then s and t are in the same connected component of J if there exist $s_1, ..., s_k \in J$ such that $ss_1 \neq s_1 s$, $s_1 s_2 \neq s_2 s_1, ..., s_{k-1}s_k \neq s_k s_{k-1}$, and $s_k t \neq t s_k$.

The following theorem indicates exactly how to detect, combinatorially, the condition of Definiton 5.7.

Theorem 5.8. *Let $\lambda \in \mathcal{C}$. The following are equivalent.*

1. *$\mathcal{P}_\lambda$ is a simple polytope.*

2. *There are exactly $|S|$ edges of $\mathcal{P}_\lambda$ meeting at λ.*

3. *$J = \{s \in S \mid s(\lambda) = \lambda\}$ has the properties*

   (a) *If $s \in S \backslash J$, and $J \not\subseteq C_W(s)$, then there is a unique $t \in J$ such that $st \neq ts$. If $C \in \pi_0(J)$ is the unique connected component of J with $t \in C$ then $C \backslash \{t\} \subseteq C$ is a setup of type $A_{l-1} \subseteq A_l$.*

   (b) *For each $C \in \pi_0(J)$ there is a unique $s \in S \backslash J$ such that $st \neq ts$ for some $t \in C$.*

13

One can list all possible subsets $J \subseteq S$ that are combinatorially smooth. We do this according to the type of the underlying simple group. The numbering of the elements of S is as follows. For types A_n, B_n, C_n, F_4, and G_2 it is the usual numbering. In these cases the end nodes are s_1 and s_n. For type E_6 the end nodes are s_1, s_5 and s_6 with $s_3 s_6 \neq s_6 s_3$. For type E_7 the end nodes are s_1, s_6 and s_7 with $s_4 s_7 \neq s_7 s_4$. For type E_8 the end nodes are s_1, s_7 and s_8 with $s_5 s_8 \neq s_8 s_5$. In each case of type E_n, the nodes corresponding to $s_1, s_2, ..., s_{n-1}$ determine the unique subdiagram of type A_{n-1}. For type D_n the end nodes are s_1, s_{n-1} and s_n. The two subdiagrams of D_n, of type A_{n-1}, correspond to the subsets $\{s_1, s_2, ..., s_{n-2}, s_{n-1}\}$ and $\{s_1, s_2, ..., s_{n-2}, s_n\}$ of S.

1. A_1.

   (a) $J = \phi$.

   A_n, $n \geq 2$. Let $S = \{s_1, ..., s_n\}$.

   (a) $J = \phi$.
   (b) $J = \{s_1, ..., s_i\}$, $1 \leq i < n$.
   (c) $J = \{s_j, ..., s_n\}$, $1 < j \leq n$.
   (d) $J = \{s_1, ..., s_i, s_j, ...s_n\}$, $1 \leq i$, $i \leq j - 3$ and $j \leq n$.

2. B_2.

   (a) $J = \phi$.
   (b) $J = \{s_1\}$.
   (c) $J = \{s_2\}$.

   B_n, $n \geq 3$. Let $S = \{s_1, ..., s_n\}$, α_n short.

   (a) $J = \phi$.
   (b) $J = \{s_1, ..., s_i\}$, $1 \leq i < n$.
   (c) $J = \{s_n\}$.
   (d) $J = \{s_1, ..., s_i, s_n\}$, $1 \leq i$ and $i \leq n - 3$.

3. C_n, $n \geq 3$. Let $S = \{s_1, ..., s_n\}$, α_n long.

   (a) $J = \phi$.
   (b) $J = \{s_1, ..., s_i\}$, $1 \leq i < n$.
   (c) $J = \{s_n\}$.
   (d) $J = \{s_1, ..., s_i, s_n\}$, $1 \leq i$ and $i \leq n - 3$.

4. D_n, $n \geq 4$. Let $S = \{s_1, ...s_{n-2}, s_{n-1}, s_n\}$.

   (a) $J = \phi$.

(b) $J = \{s_1, ..., s_i\}$, $1 \leq i \leq n - 3$.

(c) $J = \{s_{n-1}\}$.

(d) $J = \{s_n\}$.

(e) $J = \{s_1, ..., s_i, s_{n-1}\}$, $1 \leq i \leq n - 4$.

(f) $J = \{s_1, ..., s_i, s_n\}$, $1 \leq i \leq n - 4$.

5. E_6. Let $S = \{s_1, s_2, s_3, s_4, s_5, s_6\}$.

    (a) $J = \phi$.

    (b) $J = \{s_1\}$ or $\{s_1, s_2\}$.

    (c) $J = \{s_5\}$ or $\{s_4, s_5\}$.

    (d) $J = \{s_6\}$.

    (e) $J = \{s_1, s_5\}, \{s_1, s_2, s_5\}$ or $\{s_1, s_4, s_5\}$.

    (f) $J = \{s_1, s_6\}$.

    (g) $J = \{s_5, s_6\}$

    (h) $J = \{s_1, s_5, s_6\}$.

6. E_7. Let $S = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7\}$.

    (a) $J = \phi$.

    (b) $J = \{s_1\}, \{s_1, s_2\}$ or $\{s_1, s_2, s_3\}$.

    (c) $J = \{s_6\}$ or $\{s_5, s_6\}$.

    (d) $J = \{s_7\}$.

    (e) $J = \{s_1, s_6\}, \{s_1, s_2, s_6\}, \{s_1, s_2, s_3, s_6\}, \{s_1, s_5, s_6\}$, or $\{s_1, s_2, s_5, s_6\}$.

    (f) $J = \{s_6, s_7\}$.

    (g) $J = \{s_1, s_7\}$ or $\{s_1, s_2, s_7\}$.

    (h) $J = \{s_1, s_6, s_7\}, \{s_1, s_2, s_6, s_7\}$.

7. E_8. Let $S = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8\}$.

    (a) $J = \phi$.

    (b) $J = \{s_1\}, \{s_1, s_2\}, \{s_1, s_2, s_3\}$ or $\{s_1, s_2, s_3, s_4\}$.

    (c) $J = \{s_7\}$ or $\{s_6, s_7\}$.

    (d) $J = \{s_8\}$.

    (e) $J = \{s_1, s_7\}, \{s_1, s_2, s_7\}, \{s_1, s_2, s_3, s_7\}, \{s_1, s_2, s_3, s_4, s_7\},$
       $\{s_1, s_6, s_7\}, \{s_1, s_2, s_6, s_7\}, \{s_1, s_2, s_3, s_6, s_7\}$ or $\{s_1, s_2, s_5, s_6\}$.

    (f) $J = \{s_7, s_8\}$.

    (g) $J = \{s_1, s_8\}, \{s_1, s_2, s_8\}$ or $\{s_1, s_2, s_3, s_8\}$.

(h) $J = \{s_1, s_7, s_8\}, \{s_1, s_2, s_7, s_8\}$.

8. F_4. Let $S = \{s_1, s_2, s_3, s_4\}$.

   (a) $J = \phi$.
   (b) $J = \{s_1\}$ or $\{s_1, s_2\}$.
   (c) $J = \{s_4\}$ or $\{s_3, s_4\}$.
   (d) $J = \{s_1, s_4\}$.

9. G_2. Let $S = \{s_1, s_2\}$.

   (a) $J = \phi$.
   (b) $J = \{s_1\}$.
   (c) $J = \{s_2\}$.

See Corollary 3.5 of [20] for more discussion.

Associated with $J \subseteq S$ is the rational polytope $\mathcal{P}_\lambda$, where λ is chosen so that $J = \{s \in S \mid s(\lambda) = \lambda\}$. This polytope determines a projective torus embedding $X(J)$ via the "central fan construction", or more naively as

$$X(J) = \overline{[\rho_\lambda(T) \setminus \{0\}]}/K^*,$$

where ρ_λ is the irreducible representation of G with highest weight λ. $X(J)$ is independent of λ and depends only on J. The following theorem vindicates the introduction of descent systems.

Theorem 5.9. *The following are equivalent.*

1. *J is combinatorially smooth.*

2. *$X(J)$ is rationally smooth.*

3. *$\mathbb{P}(M)$ is rationally smooth.*

Indeed, Theorem 5.9 follows directly from Theorem 3.1 and Theorem 3.3.

These toric varieties $X(J)$ are of interest in there own right. The following Theorem is proved in [22].

Theorem 5.10. *Assume that $X(J)$ is rationally smooth. Then the Poincaré polynomial of $X(J)$ is*

$$P(X(J), t) = \sum_{w \in W^J} t^{2\nu(w)}.$$

where ν is as in Definition 5.5.

See Examples 4.6 and 4.9 of [22] for the details of the following two examples.

Example 5.11. Let $(W_n, S_n) = <s_1, s_2, ..., s_n> (n \geq 2)$, where $S_n = \{s_1, s_2, ..., s_n\}$ (the symmetric group) and let $J = \{s_3, s_4, ..., s_n\} \subset S_n$. Then $X(J) = X_n(J)$ is rationally smooth and

$$P(X_n(J), t) = t^{2n} + (n+2)t^{2(n-1)} + (n+2)t^{2(n-2)} + ... + (n+2)t^4 + (n+2)t^2 + 1.$$

Example 5.12. In this example we consider the root system of type B_l. Let E be a real vector space with orthonormal basis $\{\epsilon_1, ..., \epsilon_l\}$. Then

$$\Phi^+ = \{\epsilon_i - \epsilon_j \mid i < j\} \cup \{\epsilon_i + \epsilon_j \mid i \neq j\} \cup \{\epsilon_i\}, \text{ and}$$

$$\Delta = \{\epsilon_1 - \epsilon_2, ..., \epsilon_{l-1} - \epsilon_l, \epsilon_l\} = \{\alpha_1, ..., \alpha_l\}.$$

Let $S = \{s_1, s_2, ..., s_{l-1}, s_l\}$ be the corresponding set of simple reflections. Here we consider the case

$$J = \{s_1, ..., s_{l-1}\}.$$

One checks that $X(J)$ is rationally smooth. An easy calculation, as in Example 4.9 of [22], yields

$$P(X(J), t) = \sum_{w \in W^J} t^{2\nu(w)} = \sum_{A \subset \{1,...,l\}} t^{2|A|} = (1 + t^2)^l.$$

Theorem 5.10 is a fundamental ingredient in the calculation of the H-polynomal of a simple embedding. See Theorem 6.5 below.

6 Examples

In this section we calculate the H-polynomials of several types of embeddings. First we discuss all projective, semisimple rank two embeddings. Then we discuss the wonderful embedding, which was the major motivation for the entire theory of descent systems and H-polynomials. Finally we discuss rationally smooth, **simple embeddings**. These simple embeddings illustrate the role of descent systems (§5).

6.1 Semisimple Rank Two Embeddings

In this subsection we produce an explicit formula for the H-polynomial of an embedding of type A_2, C_2 and G_2. More details concerning these examples are written down in [19].

Let G be a semisimple group of type A_2, C_2 or G_2, and let ρ be a rational representation of G. As before we define X_ρ to be the $G \times G$-embedding associated with ρ. Then X_ρ is rationally smooth since any two-dimensional torus embedding is rationally smooth.

Each of these semisimple groups G has dimension $2N + 2$ where N is the length of the longest element in the Weyl group of G. Furthermore the Weyl group of G has order $2N$. One obtains that $N = 3$ for type A_2, $N = 4$ for type C_2 and $N = 6$ for type G_2.

Let B be a Borel subgroup of G. There are three cases to consider here, depending on the closed $G \times G$-orbits.

(I) All closed $G \times G$-orbits are isomorphic to $G/B \times G/B$.

(II) Exactly one closed $G \times G$-orbit is not isomorphic to $G/B \times G/B$.

(III) Exactly two closed $G \times G$-orbits are not isomorphic to $G/B \times G/B$.

Example 6.1. All closed $G \times G$-orbits are isomorphic to $G/B \times G/B$. After a simple calculation, as in Example 6.1 of [19], we obtain that the H-polynomial of X is as follows.

$$H_X(t) = [1 + (k-1)t + 2k(t^2 + \cdots + t^N) + (k-1)t^{N+1} + t^{N+2}]H(G/B)$$

where $H(G/B) = (1+t)(1 + t + \cdots + t^{N-1})$ and k is the number of closed $G \times G$-orbits.

Example 6.2. Exactly one closed $G \times G$-orbit is not isomorphic to $G/B \times G/B$. After a simple calculation, as in Example 6.2 of [19], we obtain that the H-polynomial of X is as follows.

$$H_X(t) = [t^{N+3} + kt^{N+2} + 3kt^{N+1} + (4k+1)(t^N + \cdots + t^3) + 3kt^2 + kt + 1]H(G/P)$$

where $H(G/P) = 1 + t + \cdots + t^{N-1}$ and k is the number of closed $G \times G$-orbits isomorphic to $G/B \times G/B$.

Example 6.3. Exactly two closed $G \times G$-orbits are not isomorphic to $G/B \times G/B$. After a simple calculation, as in Example 6.3 of [19], we obtain that the H-polynomial of X is as follows.

$$H_X(t) = [t^{N+3} + kt^{N+2} + (3k+1)t^{N+1} + (4k+2)(t^N + \cdots + t^3) + (3k+1)t^2 + kt + 1]H(G/P)$$

where $H(G/P) = 1 + t + \cdots + t^{N-1}$ and k is the number of closed $G \times G$-orbits isomorphic to $G/B \times G/B$.

6.2 The Wonderful Embedding

The **wonderful embedding** corresponds to the case of an embedding X_ρ where the representation ρ is irreducible with highest weight in general position. This corresponds to the situation of Theorem 5.8 where $J = \emptyset$. We single it out because it has a special significance in the theory of embeddings [9]. A semisimple monoid M is called *canonical* if $\Lambda_1 = \{e\}$, and $C_G(e)$ is a maximal torus (this is the smallest the centralizer of an idempotent can be). These monoids have been studied in detail by Putcha and the author in [13]. Any two canonical monoids M and M' have the same H-polynomial since $\mathbb{P}(M) = (M \setminus \{0\})/Z \cong \mathbb{P}(M') = (M' \setminus \{0\})/Z'$ as $G \times G$-varieties. X is related to the much-studied wonderful embedding of G/Z and we have obtained an explicit cell decomposition $X = \bigsqcup_r C_r$ of X in [17].

$$H_{\mathbb{P}(M)}(t) = \left[\sum_{u \in W} t^{l(w_0) - l(u) + |I_u|} \right] \left[\sum_{v \in W} t^{l(v)} \right]$$

where $I_u = \{s \in S \mid u < us\}$. I_u is called the *ascent set* of u. In the notation of Theorem 4.2

$$\nu(e) = |I_u|$$

where $e = ue_1u^{-1}$. Thus the Poincaré polynomial of $\mathbb{P}(M)$ is

$$P(t) = \left[\sum_{u \in W} t^{2(l(w_0) - l(u) + |I_u|)} \right] \left[\sum_{v \in W} t^{2l(v)} \right]$$

6.3   Simple Embeddings

In this section we discuss **combinatorially smooth** subsets J of S. These subsets correspond to the rationally smooth $G \times G$-embeddings X_ρ of a semisimple group G with ρ an irreducible representation. The formulas of this subsection are the culmination of the results of [19, 20, 21, 22, 23].

Recall that

$$X_\rho = \mathbb{P}(J)$$

for some unique $J \subseteq S$. In particular, X_ρ depends on J, but not ρ. See Definition 5.7 and Theorem 5.8. These embeddings are **simple embeddings** in the sense that they have exactly one closed $G \times G$-orbit. We obtain the H-polynomial of such X_ρ in terms of the augmented poset $(E_1, \leq, \{\nu_s\})$. See Definition 5.5. These simple embeddings correspond to $\mathcal{J}$-irreducible monoids. See [12] for a detailed discussion of $\mathcal{J}$-irreducible monoids. See [23] for a detailed account of the results of this section.

The result that gets us going here is the following.

Theorem 6.4. *The following are equivalent.*

   i) $J \subseteq S$ is combinatorially smooth.

   ii) $\mathbb{P}(J)$ is rationally smooth.

Theorem 6.4 follows directly from the results of Sections 3 and 5.

Let $J \subseteq S$ be combinatorially smooth and let $s \in S \setminus J$, $w \in W^J$. Then there is at most one irreducible component $C_s \subseteq J$ such that, for some $t \in J$, $st \neq ts$. Set

   a) $\delta(s) = |C_s| + 1$, and

   b) $\nu_s(w) = |\{r \in S_s^J \mid w < wr \}|$. (which, by a previous definition, equals $|A_s^J(w)|$)

Notice that $\delta(s) = 1$ if and only if $st = ts$ for all $t \in J$.

Let $w_0 \in W^J$ be the longest element (so that $l(w_0) = dim(U_J)$, where U_J is the unipotent radical of P_J.).

The following Theorem is proved in [23].

Theorem 6.5. *Let M be $\mathcal{J}$-irreducible of type $J \subseteq S$. Assume that $J \subseteq S$ is combinatorially smooth. Then the H-polynomial of $\mathbb{P}(M)$ is given by*

$$H_{\mathbb{P}(M)}(t) = \left(\sum_{w \in W^J} t^{l(w_0)-l(w)+\nu(w)} \right) \left(\sum_{v \in W^J} t^{l(v)} \right)$$

where $\nu(w) = \sum_{s \in S \setminus J} \delta(s) \nu_s(w)$ and $H(J) = \sum_{v \in W^J} t^{l(v)}$, the H-polynomial of G/P_J.

Example 6.6. Let $M = M_{n+1}(\mathbb{C})$. Then M is $\mathcal{J}$-irreducible of type $J \subset S$, where $J = \{s_2, s_3, ..., s_n\}$ and $S = S_n = \{s_1, s_2, ..., s_n\} \subset W_n$ is of type A_n $(n \geq 1)$. In this example

$S^J = \{s_1, s_2s_1, s_3s_2s_1, ..., s_n \cdots s_1\}$, and
$W^J = S^J \sqcup \{1\}$.

Write $a_i = s_i \cdots s_1$ if $i > 1$, and $a_0 = 1$. An elementary calculation yields

$S \setminus J = \{s_1\}$,
$l(a_i) = i$,
$w_0 = s_n \cdots s_1$,
$\delta(s_1) = n$,
$\nu_{s_1}(a_i) = n - i$,
$H(J) = \sum_{i=0}^{n} t^{2i}$, and
$\mathbb{P}(M) = \mathbb{P}^{(n+1)^2-1}(\mathbb{C})$.

Another elementary calculation (using Theorem 6.5) then yields

$$H_{\mathbb{P}(M)}(t) = \left(\sum_{i=0}^{n} t^{(n-i)(n+1)} \right) \left(\sum_{i=0}^{n} t^i \right) = \sum_{i=0}^{(n+1)^2-1} t^i.$$

Example 6.7. In this example we illustrate Theorem 6.5 by calculating the Poincaré polynomial of $\mathbb{P}(M)$ where M is $\mathcal{J}$-irreducible of type $J \subset S$, where $S = S_n = \{s_1, s_2, ..., s_n\} \subset W_n$ is of type A_n $(n \geq 2)$ and $J = J_n = \{s_3, s_4, ..., s_n\}$.

If $w \in W_n^J$ we can write $w = a_p b_q$ where $a_p = s_p \cdots s_1$ $(1 \leq p \leq n)$ and $b_q = s_q \cdots s_2$ $(2 \leq q \leq n)$. We also adopt the peculiar but useful convention $a_0 = 1$ and $b_1 = 1$. Thus

$$W_n^J = \{a_p b_q \mid 0 \leq p \leq n \text{ and } 1 \leq q \leq n\}$$

with uniqueness of decomposition.

Now $S \setminus J = \{s_1, s_2\}$ so that $C_{s_1} = \phi$ and $C_{s_2} = \{s_3, ..., s_n\}$. Thus,

i) $\delta(s_1) = 1$, and

ii) $\delta(s_2) = (n-2) + 1 = n - 1$.

Then, from Example 5.6,

i) $\nu_{s_1}(a_p b_q) = 1$ if $p < q$ and
$\nu_{s_1}(a_p b_q) = 0$ if $p \geq q$.

ii) $\nu_{s_2}(a_p b_q) = n - q$.

Thus, by definition,

i) $\nu(a_p b_q) = (n-1)(n-q) + 1$ if $p < q$ and

ii) $\nu(a_p b_q) = (n-1)(n-q)$ if $p \geq q$.

Finally,

i) $l(a_p b_q) = p + q - 1$, and

ii) $a_n b_n \in W^J$ is the longest element.

Thus, for $w = a_p b_q \in W^J$, we obtain by elementary calculation that

$$l(w_0) - l(w) + \nu(w) = n - p + n(n-q) + \epsilon$$

where $\epsilon = 1$ if $0 \leq p < q \leq n$, and $\epsilon = 0$ if $n \geq p \geq q \geq 1$. Thus

$$\sum_{w \in W^J} t^{l(w_0) - l(w) + m(w)} = \sum_{0 \leq p < q \leq n} t^{n-p+n(n-q)+1} + \sum_{n \geq p \geq q \geq 1} t^{n-p+n(n-q)}$$

The other factor here is

$$H(J) = \sum_{w \in W^J} t^{l(w)} = \sum_{i=1}^{n} i(t^{i-1} + t^{2n-i}).$$

Finally we obtain

$$H_{\mathbb{P}(M)}(t) = \left(\sum_{0 \leq p < q \leq n} t^{n-p+n(n-q)+1} + \sum_{n \geq p \geq q \geq 1} t^{n-p+n(n-q)} \right) \left(\sum_{i=1}^{n} i(t^{i-1} + t^{2n-i}) \right).$$

Example 6.8. In this example we consider the root system of type B_l. Let E be a real vector space with orthonormal basis $\{\epsilon_1, ..., \epsilon_l\}$. Then

$$\Phi^+ = \{\epsilon_i - \epsilon_j \mid i < j\} \cup \{\epsilon_i + \epsilon_j \mid i \neq j\} \cup \{\epsilon_i\}, \text{ and}$$

$$\Delta = \{\epsilon_1 - \epsilon_2, ..., \epsilon_{l-1} - \epsilon_l, \epsilon_l\} = \{\alpha_1, ..., \alpha_l\}.$$

Let $S = \{s_1, s_2, ..., s_{l-1}, s_l\}$ be the corresponding set of simple reflections. Here we consider the case the $\mathcal{J}$-irreducible monoid M of type

$$J = \{s_1, ..., s_{l-1}\} \subseteq S.$$

We make the following identification.

$$W^J \cong \{1 \le i_1 < i_2 < ... < i_k \le l\}$$

as follows. Given such a sequence, $1 \le i_1 < i_2 < ... < i_k \le l$, we define

$$w(\epsilon_v) = \epsilon_{i_v} \text{ for } 1 \le v \le k,$$

and

$$w(\epsilon_{k+v}) = -\epsilon_{j_v} \text{ for } 1 \le v \le l - k,$$

where $l \ge j_1 > j_2 > ... > j_{l-k} \ge 1$ (so that $\{1, ..., l\} = \{i_1, i_2, ..., i_k\} \sqcup \{j_1, j_2, ..., j_{l-k}\}$). One can check that $w \in W^J$ and that, conversely, any element of W^J is of this form.

With these identifications we let $w \in W^J$. We now recall that

$$A^J(w) = \{r \in S^J \mid w < wr\}$$

and that

$$S^J = \{s_1 \cdots s_l, s_2 \cdots s_l, ..., s_i \cdots s_l, ..., s_{l-1}s_l, s_l\}.$$

Let $w \in W^J$ correspond, as above, to $i_1 < ... < i_k$ and $j_1 > ... > j_{l-k}$. Let $r_i = s_i \cdots s_l \in S^J$. By the calculations of [20], $w < wr_i$ if and only if $i \le k$. Thus we obtain

$$A^J(w) = \{s_1 \cdots s_l, ..., s_k \cdots s_l\} = \{r \in S^J \mid w < wr\}.$$

Now we can use Theorem 6.5 above to obtain the H-polynomial of M. Let us first assemble the relevant information.

1. $S \setminus J = \{s_l\}$.

2. $\delta(s_l) = C_{s_l} + 1 = |\{s_1, ..., s_{l-1}\}| + 1 = l$.

3. If $w \in W^J$ then $\nu_{s_l}(w) = k$ where

$$w \longleftrightarrow \{1 \le i_1 < i_2 < ... < i_k \le l\}$$

   as above.

4. $\nu(w) = l\nu_{s_l}(w) = kl$.

5. $l(w_0) - l(w) = \sum_{i \in M'(w)} i$ where $M'(w) = \{i \mid w(\epsilon_j) = \epsilon_i \text{ for some } j\} = \{i_1, i_2, ..., i_k\}$, and where $w_0 \in W^J$ is the longest element (notice that $l(w_0) = l(l+1)/2$).

Collecting terms we obtain that, for $w \in W^J$,

$$l(w_0) - l(w) + \nu(w) = [\sum_{i \in M'(w)} i] + l|M'(w)| = \sum_{i \in M'(w)} (i + l).$$

After recalling some elementary generating functions, and applying Theorem 6.5, we obtain that

$$H_{\mathbb{P}(M)}(t) = \left[\Pi_{k=1}^l (1 + t^{k+l})\right] \left[\Pi_{k=1}^l (1 + t^k)\right].$$

The $\Pi_{k=1}^l(1 + t^k)$ factor here is $H(G/P_J) = \sum_{v \in W^J} t^{l(v)}$ and the $\Pi_{k=1}^l(1 + t^{k+l})$ factor is $\sum_{w \in W^J} t^{l(w_0)-l(w)+\nu(w)}$.

References

[1] V. Alexeev, M. Brion, *Stable reductive varieties II. Projective case*, Adv. Math.184(2004), 380-408.

[2] A. Bialynicki-Birula, *Some theorems on the actions of algebraic groups*, Annals of Math. 98(1973), 480-497.

[3] S. Billey, V. Lakshmibai, *Singular Loci of Schubert Varieties*, Progress in Mathematics vol. 182, Birkhauser, Boston, 2000.

[4] M. Brion, *The behaviour of the Bruhat decomposition at infinity*, Comment. Math. Helv. 73(1998), 137-174.

[5] M. Brion, *Rational smoothness and fixed points of torus actions*, Transformation Groups 4(1999), 127-156.

[6] M. Brion, *Local structure of algebraic monoids*, Semigroup Forum, to appear.

[7] M. Brion, R. Joshua, *Intersection Cohomology of Reductive Varieties*, J. Eur. Math. Soc. 6(2004), 465-481.

[8] V. I. Danilov, *The geometry of toric varieties*, Russian Math. Surveys 33(1978), 97-154.

[9] C. de Concini, C. Procesi, *Complete symmetric varieties*, Springer Lecture Notes in Mathematics 996(1983), 1-44.

[10] C. McRory, *A Characterization of Homology Manifolds*, J. London Math. Soc. 16(1977), 146-159.

[11] M. S. Putcha, *Linear algebraic monoids*, Cambridge University Press, 1988.

[12] M. S. Putcha and L. E. Renner, *The system of idempotents and lattice of J-classes of reductive algebraic monoids*, Journal of Algebra 116(1988), 385-399.

[13] M. S. Putcha and L. E. Renner, *The canonical compactification of a finite group of Lie type*, Trans. Amer. Math. Soc. 337(1993), 305-319.

[14] L. E. Renner, *Classification of semisimple algebraic monoids*, Trans. Amer. Math. Soc. 292(1985), 193-223.

[15] L. E. Renner, *Analogue of the Bruhat decomposition for algebraic monoids*, Journal of Algebra 101(1986), 303-338.

[16] L. E. Renner, *Classification of semisimple varieties*, Journal of Algebra 122(1989), 275-287.

[17] L. E. Renner, *An explicit cell decomposition of the canonical compactification of an algebraic group*, Can. Math. Bull., 46(2003), 140-148.

[18] L. E. Renner, *"Linear algebraic monoids"*, Encyclopedia of Mathematical Sciences, vol. 134, Subseries: Invariant Theory, vol. 5, Springer-Verlag, 2005.

[19] L. E. Renner, *The H-polynomial of a semisimple monoid*, J. of Algebra 319(2008), 360-376.

[20] L. E. Renner, *Descent Systems for Bruhat Posets*, J. of Alg. Comb. 29(2009), 413-435.

[21] L. E. Renner, *Rationally smooth algebraic monoids*, Semigroup Forum, to appear.

[22] L. E. Renner, *Weyl Groups, Descent Systems and Betti Numbers*, Rocky Mountain Journal of Mathematics, to appear.

[23] L. E. Renner, *The H-polynomial of an Irreducible Representation*, to appear.

[24] L. Solomon, *An introduction to reductive monoids*, "Semigroups, formal languages and groups", J. Fountain, Ed., Kluwer Academic Publishers, 1995, 295-352.

[25] T. A. Springer, *Intersection cohomology of $B \times B$ - orbit closures in group compactifications*, Journal of Algebra 258(2002), 71-111.

Lex E. Renner
Department of Mathematics
University of Western Ontario
London, N6A 5B7, Canada